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# BAYESIAN INFERENCE FOR A FAMILY BASED ON THE WEIBULL AND THE POWER SERIES DISTRIBUTIONS

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# Bayesian inference for a family based on the Weibull and the power series distributions

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**Abstract**

In this work we deal with Bayesian inference for the parameters of some distributions in the Weibull power series family. For statistical modeling purposes, this class of three parameter distributions allows great flexibility. The hazard rate function accommodates increasing, decreasing and upside down bathtub shapes. Furthermore, the density function can be bimodal. We base our inferences on the Markov chain Monte Carlo (MCMC) simulation methods. A goodness of fit diagnostic is developed using samples from the posterior distribution. Results from a simulation study aimed to assess some frequentist properties of the estimators are reported. The methodology is illustrated with a real data set.

# 1 Introduction

Distributions for modeling the life length of individuals and materials are extensively studied in the statistical literature under the headings of reliability and survival analysis. Recently, many authors concentrate on proposals based on modifications of well-known distributions, as in the works by Gupta & Kundu (2007), Carrasco *et al.* (2009), Silva *et al.* (2010), Ristić & Balakrishnan (2012), Cordeiro *et al.* (2013) to name just a few. A Bayesian approach for the modified Weibull (Lai *et al.*, 2003) was proprosed by Upadhyay & Gupta (2010). On the other side, following a different stream of research, some authors presented new lifetime distributions by compounding the exponential distribution with discrete distributions. Adamidis & Loukas (1998) and Kuş (2007) adopted the geometric and the Poisson distribution giving rise to the exponential geometric (EG) and exponential Poisson (EP) distributions, respectively. The exponential logarithmic (EL) distribution of Tahmasbi & Rezaei (2008) stems from the composition between the exponential and the logarithmic distributions. The two-parameter family proposed by Chahkandi & Ganjali

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Table 1: Some truncated at 0 distributions in the power series family.

| Distribution | $a_z$ | $C(\theta)$ | $C'(\theta)$ | $\Theta$ |
|---|---|---|---|---|
| Geometric | 1 | $\frac{\theta}{1-\theta}$ | $\frac{1}{(1-\theta)^2}$ | $(0,1)$ |
| Poisson | $1/z!$ | $e^\theta - 1$ | $e^\theta$ | $(0,\infty)$ |
| Logarithmic | $1/z$ | $-\log(1-\theta)$ | $\frac{1}{(1-\theta)}$ | $(0,1)$ |
| Binomial | $\binom{m}{z}$ | $(1+\theta)^m - 1$ | $m(1+\theta)^{m-1}$ | $(0,\infty)$ |

(2009) includes as special cases the EG, EP, EL and exponential binomial (EB) distributions. This family has decreasing failure rate (DFR) that could have sound motivation, as stressed by Chahkandi & Ganjali (2009), among others. However, it would be desirable to accommodate other behaviors of the hazard rate function.

Hence, by compounding the Weibull and the power series distributions, Morais & Barreto-Souza (2011) proposed the Weibull power series (WPS) class of distributions. Their construction runs as follows. Let $Y_1, \ldots, Y_Z$ be a random sample from the Weibull distribution with density function $f(y|\alpha,\beta) = \alpha\beta y^{\alpha-1}\exp(-\beta y^\alpha)$, for $\alpha > 0, \beta > 0$ and $y > 0$, where $Z$ follows the truncated at 0 power series distribution with probability function

$$p(z|\theta) = \frac{a_z \theta^z}{C(\theta)}, \; z = 1, 2, \ldots,$$

where $\theta \in \Theta$, $a_z > 0$ does not depend on $\theta$ and $C(\theta) = \sum_{k=1}^{\infty} a_k \theta^k$ (see Table 1). Let $X = \min(Y_1, \ldots, Y_Z)$, so that the distribution of $X|Z = z$ is Weibull with parameters $\alpha$ and $\beta z$. The marginal distribution of $X$ is termed the WPS distribution, whose density and cumulative distribution functions are given by

$$f(x|\boldsymbol{\vartheta}) = \frac{\alpha\theta\beta x^{\alpha-1} e^{-\beta x^\alpha} C'(\theta e^{-\beta x^\alpha})}{C(\theta)} \quad \text{and} \quad F(x|\boldsymbol{\vartheta}) = 1 - \frac{C(\theta e^{-\beta x^\alpha})}{C(\theta)}, \tag{1}$$

for $x > 0$, where $\boldsymbol{\vartheta} = (\alpha, \beta, \theta)'$. The identifiability of the WPS class is established in the following proposition.

**Proposition 1.1.** *The WPS class of distributions is identifiable.*

*Proof.* Let $\boldsymbol{\vartheta}_1 = (\alpha_1, \beta_1, \theta_1)'$ and $\boldsymbol{\vartheta}_2 = (\alpha_2, \beta_2, \theta_2)'$ such that $\boldsymbol{\vartheta}_1 \neq \boldsymbol{\vartheta}_2$. Suppose that $F(x; \boldsymbol{\vartheta}_1) = F(x; \boldsymbol{\vartheta}_2)$ for all $x > 0$, which from (1) implies that

$$\frac{C(\theta_2)}{C(\theta_1)} = \frac{C(\theta_2 e^{-\beta_2})}{C(\theta_1 e^{-\beta_1})} = \frac{C(\theta_2 e^{-\beta_2 2^{\alpha_2}})}{C(\theta_1 e^{-\beta_1 2^{\alpha_1}})} = \cdots. \tag{2}$$

Without loss of generality, take $\theta_2 > \theta_1$. Since $C(\cdot)$ is monotone increasing it follows that $C(\theta_2)/C(\theta_1) > 1$. For $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$, there exists $x_0$ such that $\theta_2 e^{-\beta_2 x_0^{\alpha_2}} < \theta_1 e^{-\beta_1 x_0^{\alpha_1}}$, so that $C(\theta_2 e^{-\beta_2 x_0^{\alpha_2}})/C(\theta_1 e^{-\beta_1 x_0^{\alpha_1}}) < 1$. Therefore, the equalities in (2) can not be satisfied, concluding the proof. □

Besides having as particular cases the exponential power series family of distributions, the WPS class of distributions has increasing, decreasing and upside down bathtub shaped failure rate function. Therefore, the WPS family is a more attractive class of distributions than the DFR family introduced by Chahkandi & Ganjali (2009). Of course, the wide range of applications of the WPS family is not restricted to the study of times to event data.

Here we develop inferential tools under a Bayesian viewpoint for the parameters of the WPS class of distributions proposed by Morais & Barreto-Souza (2011). Our paper unfolds as follows. In Section 2 we present the required steps to draw samples from the posterior distribution. Also, in Section 2.1 a goodness of fit diagnostic based on samples from the posterior distribution is developed. The results of a simulation study are reported in Section 3. One illustrative example with a real data set is worked out in Section 4. We end up with some remarks in Section 5.

# 2 Bayesian inference

We assume that $\alpha$, $\beta$ and $\theta$ are *a priori* independent, that is,

$$\pi(\boldsymbol{\vartheta}|\boldsymbol{h}) = \pi(\alpha|c_0, d_0)\pi(\beta|c_1, d_1)\pi(\theta|c_2, d_2), \tag{3}$$

where $\boldsymbol{h}$ denotes the vector of hyperparameters. We postulate $\pi(\alpha|c_0, d_0) = \text{Ga}(\alpha; c_0, d_0)$ and $\pi(\beta|c_1, d_1) = \text{Ga}(\beta; c_1, d_1)$, where $\text{Ga}(\cdot; c, d)$ denotes the density function of the gamma distribution with mean equal to $c/d$. For the distributions in Table 1, the prior specification for $\theta$ is $\pi(\theta|c_2, d_2) = \text{Be}(\theta; c_2, d_2)$ (geometric and logarithmic distributions) and $\pi(\theta|c_2, d_2) = \text{Ga}(\theta; c_2, d_2)$ (Poisson and binomial distributions), where $\text{Be}(\cdot; c, d)$ denotes the density function of the beta distribution. The vector of hyperparameters $\boldsymbol{h}$ is chosen to ensure vague prior knowledge. Our choice for the prior distribution enables a convenient implementation of the Gibbs sampler (Chen *et al.*, 2000), as the reason for this will become clear soon.

From (1) and (3), the posterior distribution of $\boldsymbol{\vartheta}$ is given by

$$\pi(\boldsymbol{\vartheta}|\boldsymbol{x}, \boldsymbol{h}) \propto \pi(\boldsymbol{\vartheta}|\boldsymbol{h}) \prod_{i=1}^{n} f(x_i|\boldsymbol{\vartheta}).$$

However, to ease the computations, we resort to data augmentation. The observed data $\boldsymbol{X} = (X_1, \ldots, X_n)$ is augmented by $\boldsymbol{Z} = (Z_1, \ldots, Z_n)$. Then, the density function of the complete data $(\boldsymbol{X}, \boldsymbol{Z})$, observable and unobservable variables, respectively, is given by

$$f(x, z|\boldsymbol{\vartheta}) = \frac{a_z \theta^z}{C(\theta)} \alpha\beta x^{\alpha-1} z \exp(-\beta x^\alpha z),\ x > 0,\ z = 1, 2, \ldots.$$

The likelihood function corresponding to the complete data $(\boldsymbol{X}, \boldsymbol{Z})$, with $\{(X_i, Z_i)\}$, $i = 1, \ldots, n$, conditionally independent given $\boldsymbol{\vartheta}$, has expression

$$L(\boldsymbol{\vartheta}; \boldsymbol{x}, \boldsymbol{z}) = \prod_{i=1}^{n} \frac{a_{z_i} \theta^{z_i}}{C(\theta)} \alpha\beta x_i^{\alpha-1} z_i \exp(-\beta x_i^\alpha z_i). \tag{4}$$

After combining the likelihood function in (4) with the prior distribution, the joint posterior distribution of $\boldsymbol{\vartheta}$ results to be $\pi(\boldsymbol{\vartheta}|\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{h}) \propto \pi(\alpha|c_0, d_0)\ \pi(\beta|c_1, d_1)$ $\pi(\theta|c_2, d_2)L(\boldsymbol{\vartheta}; \boldsymbol{x}, \boldsymbol{z})$. Taking into account the prior distribution in (3) and the likelihood function in (4), the the full conditional distributions turn out to be

$$\pi(z_i|x_i, \boldsymbol{\vartheta}) \propto z_i \exp(-\beta x_i^\alpha z_i) a_{z_i} \theta^{z_i}, \ \ z_i = 1, 2, \ldots, \tag{5}$$

$$\pi(\alpha|\beta, \boldsymbol{x}, \boldsymbol{z}, c_0, d_0) \propto \alpha^n \Big(\prod_{i=1}^n x_i\Big)^{\alpha-1} \exp\Big(-\beta \sum_{i=1}^n x_i^\alpha z_i\Big) \pi(\alpha|c_0, d_0), \tag{6}$$

$$\pi(\beta|\alpha, \boldsymbol{x}, \boldsymbol{z}, c_1, d_1) = \mathrm{Ga}(\beta; c_1 + n, d_1 + \sum_{i=1}^n x_i^\alpha z_i) \tag{7}$$

$$\text{and} \quad \pi(\theta|\boldsymbol{z}, c_2, d_2) \propto \frac{\theta^{\sum_{i=1}^n z_i}}{\{C(\theta)\}^n} \pi(\theta|c_2, d_2). \tag{8}$$

Notice that in (5), since $\sum_{k=1}^\infty a_k \theta^k$ is a convergent series and $k \exp(-k\beta x^\alpha)$ $\leqq \exp(-\beta x^\alpha)$, for $k \geqq 1$, the series $\sum_{k=1}^\infty k \exp(-k\beta x^\alpha)$ also converges, so that this conditional distribution is a member of the power series family. The probability mass function in (5) is given in the following proposition.

**Proposition 2.1.** *The full conditional distribution in (5) has expression*

$$\pi(z_i|x_i, \boldsymbol{\vartheta}) = \frac{z_i \exp\{-\beta x_i^\alpha (z_i - 1)\} a_{z_i} \theta^{z_i - 1}}{C'(\theta e^{-\beta x_i^\alpha})}, \ \ z_i = 1, 2, \ldots. \tag{9}$$

*Proof.* According to Johnson *et al.* (2005, Section 2.2.1), it can be shown that the moment generating function (mgf) of $Z$ evaluated at $t$ is $C(\theta e^t)/\{C(\theta) + b_0\}$, where $b_0 = 0$ for the logarithmic distribution and $b_0 = a_0$ for the remaining distributions in Table 1. Moreover, $E(Z) = \theta d \log\{C(\theta)\}/d\theta$. The normalizing constant in (5) is computed in two steps. Firstly, using the mgf of $Z$, we can write

$$\pi(z_i|x_i, \boldsymbol{\vartheta}) \propto z_i \frac{\exp(-\beta x_i^\alpha z_i) a_{z_i} \theta^{z_i}}{C(\theta e^{-\beta x_i^\alpha})} = z_i \frac{a_{z_i}^* \theta^{z_i}}{C_i^*(\theta)}, \ \ z_i = 1, 2, \ldots, \tag{10}$$

where $a_{z_i}^* = \exp(-\beta x_i^\alpha z_i) a_{z_i}$ and $C_i^*(\theta) = C(\theta e^{-\beta x_i^\alpha})$. Secondly, computing the expectation from (10) we obtain

$$\theta \frac{d}{d\theta} \log\{C_i^*(\theta)\} = \frac{\theta}{C(\theta e^{-\beta x_i^\alpha})} e^{-\beta x_i^\alpha} C'(\theta e^{-\beta x_i^\alpha})$$

as the normalizing constant for (10), which lead us to the result in (9). □

Samples from $Z$ in (9) are drawn by applying the rejection method (Devroye, 1986). In (6), the distribution is log-concave and the sampling is straightforward with the adaptive rejection method (Gilks & Wild, 1992; Wild & Gilks, 1993). We emphasize that the distribution of $\beta$ in (7) is the same whichever the distribution in Table 1.

For the geometric distribution, $\theta$ in (8) is sampled from a $\mathrm{Be}(c_2 + \sum_{i=1}^n z_i - n, d_2 + n)$ distribution, whereas the remaining distributions in Table 1 require Metropolis steps. Following Chen *et al.* (2000, Section 2.2), first we make the transformation $\xi = \log(\theta)$, for the Poisson and binomial distributions, and $\xi = \log\{\theta/(1-\theta)\}$, for the logarithmic distribution. Then, we derive $\ell(\xi) \equiv \log\{\pi(\xi|\boldsymbol{z}, c_2, d_2)\}$ from (8). Up to a constant, we have that

$$\ell(\xi) = (c_2 + \textstyle\sum_{i=1}^n z_i)\xi - d_2 e^{\xi} - n\log\{\exp(e^{\xi}) - 1\},$$
$$\ell(\xi) = (c_2 + \textstyle\sum_{i=1}^n z_i)\xi - (c_2 + d_2 + \sum_{i=1}^n z_i)\log(1+e^{\xi}) - n\log\{\log(1+e^{\xi})\}$$
$$\text{and}\quad \ell(\xi) = (c_2 + \textstyle\sum_{i=1}^n z_i)\xi - d_2 e^{\xi} - n\log\{(1+e^{\xi})^m - 1\},$$

for the Poisson, logarithmic and binomial distributions, respectively. A proposal value $\xi^* \sim \mathrm{N}(\widehat{\xi}, \widehat{\sigma}^2_{\widehat{\xi}})$ is generated, where $\widehat{\xi}$ maximizes $\ell(\xi)$ and $\widehat{\sigma}^2_{\widehat{\xi}}$ is minus the inverse of $d^2\ell(\xi)/d\xi^2$ evaluated at $\xi = \widehat{\xi}$, where

$$\frac{d^2\ell(\xi)}{d\xi^2} = -d_2 e^{\xi} + n\frac{\exp(e^{\xi}+\xi)\{e^{\xi} - \exp(e^{\xi}) + 1\}}{\{\exp(e^{\xi}) - 1\}^2},$$

$$\frac{d^2\ell(\xi)}{d\xi^2} = -(c_2 + d_2 + \sum_{i=1}^n z_i)\frac{e^{\xi}}{(1+e^{\xi})^2} - n\frac{e^{\xi}\{\log(1+e^{\xi}) - e^{\xi}\}}{\{(1+e^{\xi})\log(1+e^{\xi})\}^2}$$

$$\text{and}\quad \frac{d^2\ell(\xi)}{d\xi^2} = -d_2 e^{\xi} + mn\frac{(1+e^{\xi})^{m-2}e^{\xi}\{me^{\xi} - (1+e^{\xi})^m + 1\}}{\{(1+e^{\xi})^m - 1\}^2},$$

for the Poisson, logarithmic and binomial distributions, respectively. The maximization of $\ell(\xi)$ can be carried out with the Nelder-Mead algorithm provided by O'Neill (1971). A move from $\xi$ to $\xi^*$ is done with probability

$$\min\left\{\frac{\pi(\xi^*|\cdot)\phi\big((\xi - \widehat{\xi})/\widehat{\sigma}_{\widehat{\xi}}\big)}{\pi(\xi|\cdot)\phi\big((\xi^* - \widehat{\xi})/\widehat{\sigma}_{\widehat{\xi}}\big)}, 1\right\}, \tag{11}$$

where $\phi(\cdot)$ denotes the standard normal probability density function.

With respect to the DFR family in Chahkandi & Ganjali (2009), we set $\alpha = 1$ and skip the step in (6). The MCMC computations were implemented using the FORTRAN language. We have also written BUGS codes in OpenBUGS (Thomas *et al.*, 2006) for the distributions in Table 1. The computational codes are available on request from the authors.

## 2.1 Model assessment

In this section a device for the assessment of model fitting is presented. Taking into account the cumulative distribution function in (1), it follows that

$$F(X_i, \boldsymbol{\vartheta}) = 1 - \frac{C(\theta e^{-\beta X_i^{\alpha}})}{C(\theta)} \overset{\text{iid}}{\sim} \text{Uniform}(0,1),\ i = 1, \dots, n.$$

Therefore, under the postulated model, $Q(\boldsymbol{X}, \boldsymbol{\vartheta}) = -\sum_{i=1}^n \log\{F(X_i, \boldsymbol{\vartheta})\}$ is distributed as a $\mathrm{Ga}(n, 1)$ random variable. The pivotal quantity $Q(\boldsymbol{X}, \boldsymbol{\vartheta})$ is a key element for model checking. If $\boldsymbol{\vartheta}_0$ denotes the data generating value of $\boldsymbol{\vartheta}$ and $\boldsymbol{\vartheta}_{\text{post}}$ is drawn from the posterior distribution of $\boldsymbol{\vartheta}$ given $\boldsymbol{X}$, Johnson (2007) proved that $Q(\boldsymbol{X}, \boldsymbol{\vartheta}_0)$ and $Q(\boldsymbol{X}, \boldsymbol{\vartheta}_{\text{post}})$ have the same distribution. As recommended by Johnson (2007), a useful device for model assessment can be based on graphical comparisons of the posterior distribution of this pivotal quantity to its reference distribution. This graphical diagnostic may reveal model inadequacy.

# 3 A simulation study

In this section we present the results of a simulation study. Our study comprises the exponential logarithmic (EL), Weibull geometric (WG) and Weibull Poisson (WP) distributions. Some frequentist properties of the Bayesian estimators are assessed. The hyperparameters in (3) were set at $c_0 = d_0 = 0.01$, $c_1 = d_1 = 0.01$ and $c_2 = d_2 = 1$, whereas $c_2 = d_2 = 0.01$ for the WP distribution. For each replication, after discarding the first 1000 iterations of the Gibbs sampler, we used 30000 iterations with thinning equal to 10, thus obtaining 3000 samples for each parameter. In Table 2 are shown some summaries from 500 replications. The posterior mode estimate is the half sample mode (Bickel & Frühwirth, 2006) available in the modesst package (Poncet, 2012) in R (R Core Team, 2013). The scenario for the WP distribution resembles the conditions found in our example in Section 4.

In general, the average of the posterior absolute deviations (MAD) and the mean absolute error of the posterior estimates (MAE) do not differ so much. The biases of the posterior means, medians and modes are negligible in most of the entries in Table 2. For the parameter $\theta$, under the scenarios in our study, there is a gain in terms of bias when the posterior mode is taken as the point estimate.

Except for the parameter $\theta$ in the EL distribution, the coverage probabilities of the 95% highest posterior density (HPD) intervals differ from the nominal value by at most 2%. Since in this case the sample size is $n = 300$, this result tells us that larger sample sizes are required to reach a better approximation to the nominal value, as we can see for the WG and WP distributions. The HPD intervals were estimated following the steps described in Chen *et al.* (2000, Section 7.3.1).

# 4 Example

In this section we work out an example with the rent data set available in the gamlss.data package (Stasinopoulos & Rigby, 2012) in R. The data set comprises 1969 observations on the monthly net rent values in Munich, in German Marks. The survey was performed in April 1993. When running the Gibbs sampler, after discarding the first 2000 iterations we used 500000 iterations with thinning equal to 50, which means 10000 samples for the posterior computations.

Figure 1 shows the histogram of the posterior samples of $Q(\boldsymbol{X}, \boldsymbol{\vartheta})$ and $\text{Ga}(\cdot; 1969, 1)$ density function (left), as well as the empirical cumulative distribution function (ecdf) and WG, WP and WL distributions fitted to the data (right). The overall pattern of the plots in Figure 1(left) suggests that the best fit is achieved with the WG distribution. In fact, the WG distribution fitted to the data in Figure 1(a, right) matches more closely the ecdf than the WP and WL distributions.

The convergence of the chains was monitored by the Geweke's criterion (Geweke, 1992) and graphical inspection of the chains. The results in Table 3 (last column) and Figure 2 indicate the convergence of the chains.

In Table 3 we report the posterior summaries for the parameters of the WG distribution. The posterior medians and modes are close to the posterior means, therefore they were omitted in Table 3. It is important to stress that the estimates

Table 2: Posterior estimates from 500 replications (True: true value of the parameter, Est.: average of the posterior means, medians and modes, MAD: average of the posterior absolute deviations, MAE: mean absolute error of the posterior estimates and CP: coverage probability of the 95% HPD interval).

| | Parameter | True | Est. | MAD | MAE | CP |
|---|---|---|---|---|---|---|
| Exponential | $\beta$ | 1.5 | 1.47 | 0.107 | 0.0928 | 0.968 |
| logarithmic | | | 1.48 | 0.107 | 0.0932 | |
| $n = 300$ | | | 1.48 | 0.110 | 0.0967 | |
| | $\theta$ | 0.3 | 0.32 | 0.13 | 0.090 | 0.978 |
| | | | 0.32 | 0.13 | 0.10 | |
| | | | 0.30 | 0.15 | 0.17 | |
| Weibull | $\alpha$ | 1.5 | 1.49 | 0.0523 | 0.0532 | 0.942 |
| geometric | | | 1.49 | 0.0523 | 0.0528 | |
| $n = 1000$ | | | 1.49 | 0.0531 | 0.0533 | |
| | $\beta$ | 2 | 2.08 | 0.256 | 0.257 | 0.938 |
| | | | 2.08 | 0.256 | 0.256 | |
| | | | 2.06 | 0.262 | 0.259 | |
| | $\theta$ | 0.8 | 0.77 | 0.051 | 0.052 | 0.948 |
| | | | 0.78 | 0.051 | 0.049 | |
| | | | 0.79 | 0.053 | 0.045 | |
| Weibull | $\alpha$ | 2.7 | 2.67 | 0.0532 | 0.0562 | 0.930 |
| Poisson | | | 2.67 | 0.0521 | 0.0553 | |
| $n = 1500$ | | | 2.68 | 0.0546 | 0.0542 | |
| | $\beta$ | 0.008 | 0.000846 | 0.000180 | 0.000166 | 0.934 |
| | | | 0.000807 | 0.000160 | 0.000159 | |
| | | | 0.000741 | 0.000191 | 0.000170 | |
| | $\theta$ | 3.9 | 4.42 | 0.981 | 0.966 | 0.934 |
| | | | 4.23 | 0.955 | 0.939 | |
| | | | 3.96 | 1.04 | 1.01 | |

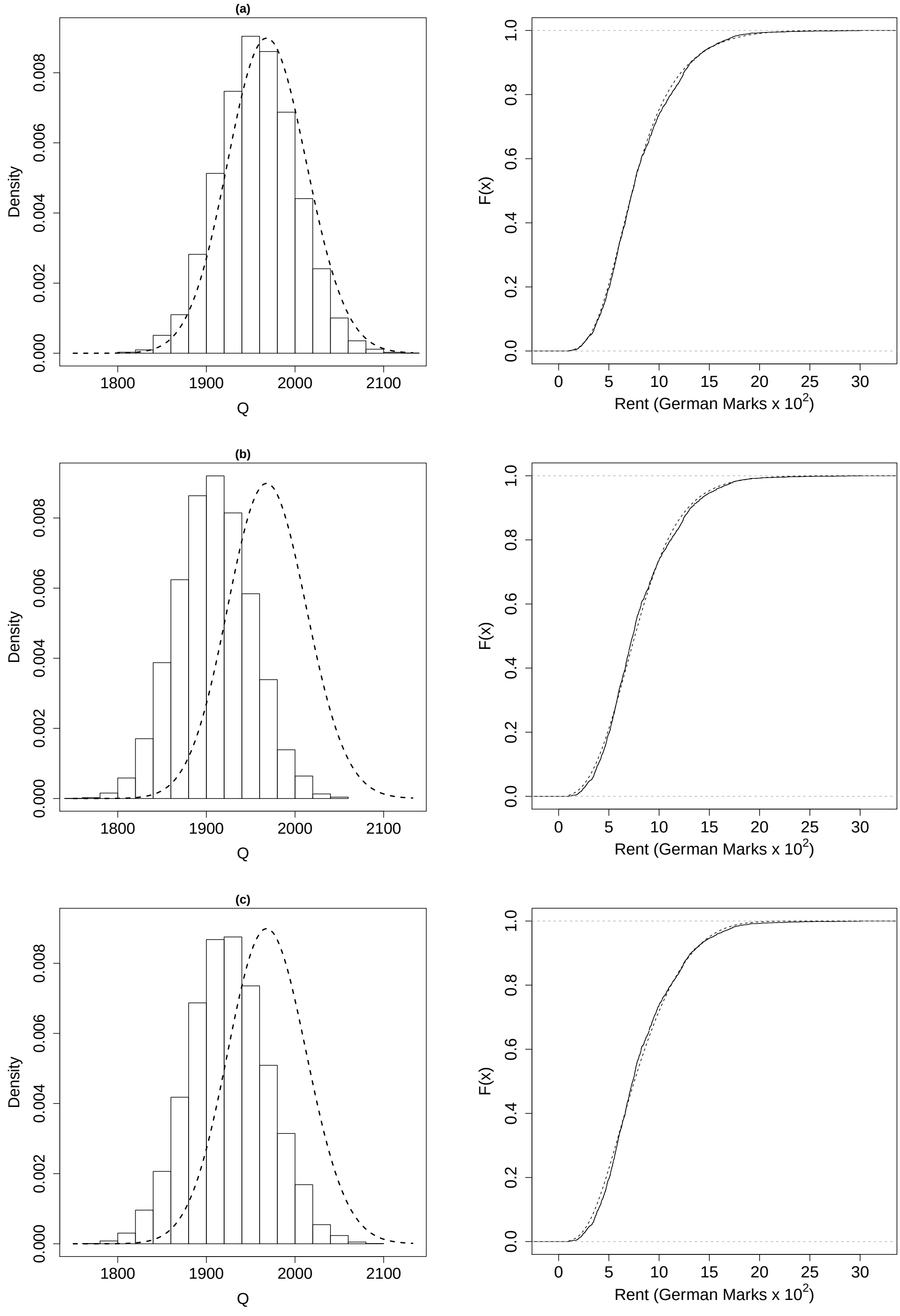


Figure 1: Left. Histogram of the posterior samples of $Q(\boldsymbol{X}, \boldsymbol{\vartheta})$ and $\text{Ga}(\cdot; 1969, 1)$ density function. Right. Empirical cumulative distribution function (solid) and fitted distributions (dashed). (a) WG, (b) WP and (c) WL distributions.

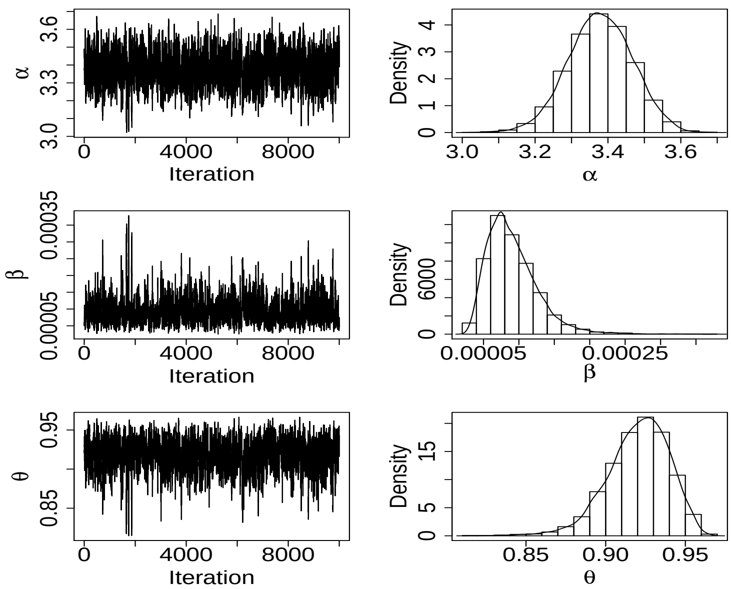


Figure 2: Trace plots of the chains, histograms and approximate marginal posterior density functions for the parameters of the WG distribution.

Table 3: Posterior summaries for the parameters of the WG distribution.

| Parameter | Mean | Standard deviation | 95% HPD interval | $p$-value (Geweke) |
|---|---|---|---|---|
| $\alpha$ | 3.38 | 0.0883 | (3.21, 3.56) | 0.535 |
| $\beta$ | 0.0000907 | 0.0000364 | (0.0000330, 0.000159) | 0.571 |
| $\theta$ | 0.920 | 0.0195 | (0.885, 0.957) | 0.615 |

of $\alpha$, both point and intervalar, are distant from 1, suggesting that the Weibull distribution does not yield a good fit. This is in agreement with graphical checks (not shown) for the Weibull distribution fit.

# 5 Conclusion

In this paper we deal with a three parameter family of distributions recently introduced by Morais & Barreto-Souza (2011). This family has attractive properties. Our proposal is based on data augmentation and MCMC simulation methods. A detailed description of the Gibbs sampler for Bayesian inference is given. Moreover, we present a diagnostic tool for model checking easily implemented as a by-product of the chains generated by the Gibbs sampler.

The role of the hyperparameters in the prior distributions was assessed. Under different degrees of vagueness of the prior distribution in (3), the differences are not important when compared to the results reported in Sections 3 and 4.

Results of simulation studies with some distributions of the DFR family were reported, for example, by Tahmasbi & Rezaei (2008) and Chahkandi & Ganjali

(2009) under a frequentist viewpoint. However, neither these authors nor Morais & Barreto-Souza (2011) pay attention to the coverage probability of the asymptotic confidence intervals based on normal approximations.

We are now working in an extension of the proposed methodology to deal with right censored data.

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